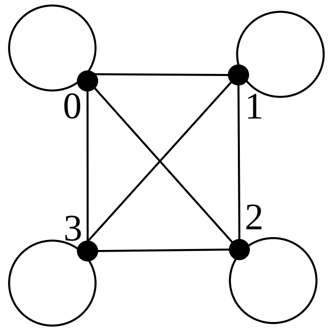
Task. DOMINO

Solution



Let us consider each of the numbers from 0 to M as a vertex of a multigraph, and the piece with u and v dots on its halves as an edge of the multigraph. The multigraph for $M = 3$ is shown on the Figure. Our task is to split the edges of the obtained after deleting of some edges multigraph in chains (paths) in such way that each edge is a part of exactly one chain. This leads as to the different kind of Euler traversals of a multigraph.

Euler circuit (and *Euler path,* respectively) in multigraph is such circuit (path) that pass through each edge exactly once. Let us resume the important results connected with the Euler traversals of multigraphs.

Lemma. In each multigraph the number of the vertices of odd degree is even.

Proof of this fact is trivial.

Theorem. The edges of connected multigraph could be ordered in Euler circuit if and only if each vertex of the multigraph is of even degree.

The proof of the Theorem is based on the fact the Euler circuit in this case has to “enter” each vertex by not traversed edge and to “leave” the vertex by not traversed edge too.

Corollary 1. The edges of connected multigraph could be ordered in Euler path if and only if two of the vertices of the multigraph are of odd degree and each other vertex is of even degree.

The proof of the Corollary consist of appending an edge between the two vertices of odd degree, constructing the obtained Euler circuit and then, after deleting of the extra edge, an Euler path from one of the vertices of odd degree to the other is obtained.
Let us apply the same construction to multigraph with 2k vertices of odd degree – to append k extra edges, each of them linking two vertices of odd degree. Now we could construct an Euler circuit in the obtained multigraph and then to delete the extra edges. The circuit will fall apart k paths and each edge will be in exactly one path. Is this splitting minimal? It is obvious that any path that start in a vertex of odd degree and do not repeat edges will stop in a vertex of odd degree. So number of chains is at least k. Because our construction generates exactly k chains it is minimal.

To solve the task for the maximal number of points one more observation is necessary. After deleting pieces of the set the obtained multigraph **could be not connected**. That is why the correct solution has to find the connected components of the multigraph first and then to construct for each component the corresponding splitting. Components with 0 vertices of odd degree will be covered by a single chain, and each component with 2k vertices of odd degree – with k chains.

Some technical problem could be that if in case of 2k vertices of odd degree traversal starts from vertex of even degree then one of the chains will be split in two in the two ends of the traversal. That is why traversal has to start always from vertex of odd degree.

Kr. Manev